Combinatorics at $\aleph_{\omega+1}$ in Prikry-type extensions

Ryan Holben

Department of Mathematics
University of California Irvine

2013 Joint Mathematics Meeting

Overview

1. Background
   - Large Cardinals
   - Singular Cardinal Combinatorics
   - Square
2. Our Results
   - Main Results
3. Future Work

Cofinality

Definition

For an infinite limit ordinal α, we define the **cofinality**:

$cf(\alpha) =$ the least limit ordinal β such that there is an increasing

β-sequence $\langle \alpha_\xi \mid \xi < \beta \rangle$ and $\lim_{\xi \to \beta} \alpha_\xi = \alpha$

- A cardinal κ is **regular** iff $cf(\kappa) = \kappa$.
- A cardinal κ is **singular** iff $cf(\kappa) < \kappa$.

Cofinality

Definition

For an infinite limit ordinal α, we define the **cofinality**:

$$cf(\alpha) = \text{the least limit ordinal } \beta \text{ such that there is an increasing}$$

$$\beta\text{-sequence } \langle \alpha_\xi \mid \xi < \beta \rangle \text{ and } \lim_{\xi \to \beta} \alpha_\xi = \alpha$$

- A cardinal κ is **regular** iff $cf(\kappa) = \kappa$.
- A cardinal κ is **singular** iff $cf(\kappa) < \kappa$.

Cofinality

Definition

For an infinite limit ordinal α, we define the **cofinality**:

$$cf(\alpha) = \text{the least limit ordinal } \beta \text{ such that there is an increasing}$$

$$\beta\text{-sequence } \langle \alpha_\xi \mid \xi < \beta \rangle \text{ and } \lim_{\xi \to \beta} \alpha_\xi = \alpha$$

- A cardinal κ is **regular** iff $cf(\kappa) = \kappa$.
- A cardinal κ is **singular** iff $cf(\kappa) < \kappa$.

Large Cardinals

- We may postulate the existence of large cardinals.

Common Large Cardinals

- Measurable, Weakly Compact, Supercompact, Woodin, etc.

- In a model of "ZFC + LC" or "ZFC + $\neg$LC" what principles must hold?
- We use large cardinals to measure the **consistency strength** of various mathematical concepts.

Large Cardinals

- We may postulate the existence of large cardinals.

Common Large Cardinals

- Measurable, Weakly Compact, Supercompact, Woodin, etc.

- In a model of "ZFC + LC" or "ZFC + ¬LC" what principles must hold?

- We use large cardinals to measure the **consistency strength** of various mathematical concepts.

Large Cardinals

- We may postulate the existence of large cardinals.

Common Large Cardinals

- Measurable, Weakly Compact, Supercompact, Woodin, etc.

- In a model of "ZFC + LC" or "ZFC + $\neg$LC" what principles must hold?

- We use large cardinals to measure the **consistency strength** of various mathematical concepts.

Large Cardinals

- We may postulate the existence of large cardinals.

Common Large Cardinals

- Measurable, Weakly Compact, Supercompact, Woodin, etc.

- In a model of "ZFC + LC" or "ZFC + $\neg$LC" what principles must hold?
- We use large cardinals to measure the **consistency strength** of various mathematical concepts.

Singular Cardinal Combinatorics

Some Interesting Combinatorial Principles

- SCH
- Tree Property
- Square principles

Singular Cardinal Combinatorics

Some Interesting Combinatorial Principles

- SCH
- Tree Property
- Square principles

Singular Cardinal Combinatorics

Some Interesting Combinatorial Principles

- SCH
- Tree Property
- Square principles

Tree Property

Definition

We say that κ has the **tree property** iff every κ tree has a cofinal branch.

- The tree property at κ^+ and $\square(\kappa^+)$ cannot both hold in the same model.

Tree Property

Definition

We say that κ has the **tree property** iff every κ tree has a cofinal branch.

- The tree property at κ^+ and $\square(\kappa^+)$ cannot both hold in the same model.

Square

Definition

We say that $\vec{C} = \langle C_\xi \mid \xi \in \lim(\kappa, \kappa^+) \rangle$ is a $\square(\kappa^+)$ sequence iff :

1. $\forall \xi \; C_\xi$ is club in ξ
2. (Coherency) $\forall \xi$ if $\beta \in lim(C_\xi)$ then $C_\xi \cap \beta = C_\beta$
3. (No thread) There is no club D in κ^+ that coheres with our sequence.

1. We say "$\square(\kappa^+)$ holds" if there exists a $\square(\kappa^+)$ sequence.

1. Other related principles include: $\square_\kappa$, $\square_{\kappa,\alpha}$, global square, etc.

Square

Definition

We say that $\vec{C} = \langle C_\xi \mid \xi \in \lim(\kappa, \kappa^+) \rangle$ is a $\square(\kappa^+)$ sequence iff :

1. $\forall \xi \; C_\xi$ is club in ξ
2. (Coherency) $\forall \xi$ if $\beta \in lim(C_\xi)$ then $C_\xi \cap \beta = C_\beta$
3. (No thread) There is no club D in κ^+ that coheres with our sequence.

1. We say "$\square(\kappa^+)$ holds" if there exists a $\square(\kappa^+)$ sequence.
1. Other related principles include: $\square_\kappa$, $\square_{\kappa,\alpha}$, global square, etc.

Square

Definition

We say that $\vec{C} = \langle C_\xi \mid \xi \in \lim(\kappa, \kappa^+) \rangle$ is a $\square(\kappa^+)$ sequence iff :

1. $\forall \xi \; C_\xi$ is club in ξ
2. (Coherency) $\forall \xi$ if $\beta \in lim(C_\xi)$ then $C_\xi \cap \beta = C_\beta$
3. (No thread) There is no club D in κ^+ that coheres with our sequence.

1. We say "$\square(\kappa^+)$ holds" if there exists a $\square(\kappa^+)$ sequence.

1. Other related principles include: $\square_\kappa$, $\square_{\kappa,\alpha}$, global square, etc.

A First Result

Theorem

$Con(ZFC + \exists \text{ a certain type of } SC) \rightarrow Con(ZFC + \neg\square(\aleph_{\omega+1}))$

Current Work

Theorem

$Con(ZFC + \exists\kappa\ SBC\ +\ meas.) \rightarrow Con(ZFC + \square_{\aleph_\omega,2} + \neg\square_{\aleph_\omega})$

Future Work

1. Consistency of $\square_{\aleph_\omega, n+1} + \neg\square_{\aleph_\omega, n}$
2. Consistency of global square at singular cardinals.
    1. Open problem from a paper of Cummings & Friedman

For Further Reading I

J. Cummings and S. Friedman.
$\square$ on the singular cardinals,
The Journal of Symbolic Logic, 2008.